Optimal Redistribution with a Shadow Economy: Online Appendix

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October 5, 2020

Abstract

The supplementary material contains the derivation of the optimal tax formula using the mechanism design approach (Appendix A), the Pareto efficiency test of the actual tax schedule in Colombia (Appendix B) and a simple model of welfare decomposition with analytical comparative statics (Appendix C).

A. Derivation of the optimal tax rates

Below we derive the optimal tax rates in terms of model primitives using the mechanism design approach, i.e. by perturbing an allocation directly subject to the incentive-compatibility constraints. Then we define the sufficient statistics used to derive the optimal tax rates in the main text and show the equivalence between the sufficient statistics approach from the main text and the mechanism design approach.

A.1. A mechanism design approach

Consider an incentive-compatible allocation (y^f, T). By Corollary 1 from Milgrom and Segal (2002) the indirect utility function $V(y^f(\theta,\kappa), T, \theta, \kappa)$ is differentiable with respect to θ almost everywhere. The derivative, by the local incentive-compatibility constraints, is given by

$$\frac{d}{d\theta}V(y^f(\theta,\kappa),T,\theta,\kappa) = \left(\rho^f(\theta)\frac{y^f(\theta,\kappa)}{w^f(\theta)} + \rho^s(\theta)\frac{y^s(\theta,\kappa)}{w^s(\theta)}\right) v'\left(\frac{y^f(\theta,\kappa)}{w^f(\theta)} + \frac{y^s(\theta,\kappa)}{w^s(\theta)}\right)$$
$$\equiv V_\theta(y^f(\theta,\kappa),T,\theta,\kappa),$$

where $\rho^x(\theta) \equiv w^x_\theta(\theta)/w^x(\theta)$ stands for the productivity growth rate in sector $x \in \{f, s\}$. From now on we will suppress for brevity the dependence of V and V_θ on the allocation. Hence, we can represent the indirect utility function in the integral form

$$V(\theta, \kappa) = V(0, \kappa) + \int_0^\theta V_\theta(\theta', \kappa) d\theta'. \tag{1}$$

Take some high-cost worker (θ, ∞). We will derive the optimality condition by perturbing formal income of this worker by small dy^f and adjusting the tax paid such that the utility level $V(\theta, \infty)$ is unchanged. This perturbation affects the slope $V_\theta(\theta, \infty)$, which in turn implies via equation (1) a uniform shift of utility levels of all high-cost types above.

Moreover, since all agents face the same tax schedule, we need to adjust the allocation of the low-cost workers as well. We can distinguish three cases. First, the distorted shadow workers can respond by marginally decreasing formal income. Second, the distorted shadow workers can respond by jumping to a discretely lower formal income level. These two cases have identical fiscal impact and lead to the same optimal tax formula. Finally, when $y^f(\theta, \infty) > y^f(\overline{\theta}, 0)$, all the shadow workers have lower formal income and hence are unaffected by the perturbation.

As mentioned in the main text, we assume that the monotonicity constraints on the formal income schedules and the constraint 3 from Proposition 2 are not binding. Both assumptions can be verified ex post and in all our applications these constraints were slack. Furthermore, we can ignore constraints 4 and 5 from Proposition 2 while deriving the optimality conditions. That is because we start from an incentive-compatible allocation and, as we show below, we do not need to keep track of the size of the intensive margin responses of shadow workers to calculate the implied fiscal loss.

Distortion of formal workers. A formal income perturbation dy^f affects the utility of type (θ, ∞) by $\left(1 - \frac{v'(n(\theta,\infty))}{w^f(\theta)}\right) dy^f$, or equivalently by $T'(y^f(\theta, \infty))dy^f$. We need to adjust the total tax paid by the same amount such that the utility level stays constant. The fiscal impact of doing so is

$$T'(y^f(\theta, \infty))(1 - G_\theta(\tilde{\kappa}(\theta)))f(\theta)dy^f. \tag{2}$$

The impact of this perturbation on the slope of the utility schedule is

$$dV_\theta(\theta, \infty) = \rho^f(\theta)\left(1 - T'(y^f(\theta, \infty))\right)\left(1 + \frac{1}{\varepsilon(\theta, \infty)}\right) dy^f, \tag{3}$$

where $\varepsilon(\theta, \kappa) \equiv \frac{v'(n(\theta,\kappa))}{n(\theta,\kappa)v''(n(\theta,\kappa))}$ is the elasticity of labor supply and $n(\theta, \kappa)$ is the total labor supply of agent (θ, κ). Hence, a perturbation that leads to a change of slope

$dV_\theta(\theta, \infty)$ implies a change in tax revenue from the formal workers by

$$\frac{T'(y^f(\theta,\infty))}{1-T'(y^f(\theta,\infty))}\left(1+\frac{1}{\varepsilon(\theta,\infty)}\right)^{-1}\frac{1}{\rho^f(\theta)}(1-G_\theta(\tilde{\kappa}(\theta)))f(\theta)dV_\theta(\theta,\infty). \tag{4}$$

Distortion of shadow workers. Let's consider the case of $y^f(\theta, \infty) \leq y^f(\overline{\theta}, 0)$, otherwise there is no tax loss from the shadow workers. The mapping $\tilde{s}(\theta) \equiv \min\{\theta' \in [\overline{\theta}, \underline{\theta}]$ s.t. $y^f(\theta', 0) \geq y^f(\theta, \infty)\}$ indicates which shadow worker is distorted by the perturbation of formal income of the high-cost worker with a productivity type θ. First, suppose that $y^f(\tilde{s}(\theta), 0) = y^f(\theta, \infty)$, so that the distorted shadow worker has the same formal income as the distorted formal worker. A perturbation of formal income dy_2^f affects the utility level of $(\tilde{s}(\theta), 0)$-type worker by $\left(1 - \frac{w^s(\tilde{s}(\theta))}{w^f(\tilde{s}(\theta))}\right) dy_2^f = T'(y^f(\tilde{s}(\theta), 0))dy_2^f$. We need to adjust the tax paid by the same amount, which affects the resource constraint by

$$T'(y^f(\tilde{s}(\theta),0))G_{\tilde{s}(\theta)}(\tilde{\kappa}(\tilde{s}(\theta)))f(\tilde{s}(\theta))dy_2^f. \tag{5}$$

The slope of the utility schedule of low-cost workers changes by

$$dV_\theta(\tilde{s}(\theta),0) = \left(\rho^f(\tilde{s}(\theta)) - \rho^s(\tilde{s}(\theta))\right)\frac{w^s(\tilde{s}(\theta))}{w^f(\tilde{s}(\theta))}dy_2^f. \tag{6}$$

The perturbation needs to respect the common tax schedule at higher formal incomes - the slopes of $V(\theta, \infty)$ and $V(\tilde{s}(\theta), 0)$ have to change by the same amount, which can be achieved by appropriately adjusting dy_2^f. Then, by using the first-order condition of workers $(\tilde{s}(\theta), 0)$, we can express the tax loss as

$$\frac{w^f(\tilde{s}(\theta)) - w^s(\tilde{s}(\theta))}{w^s(\tilde{s}(\theta))}\frac{G_{\tilde{s}(\theta)}(\tilde{\kappa}(\theta))f(\tilde{s}(\theta))}{\rho^f(\tilde{s}(\theta)) - \rho^s(\tilde{s}(\theta))))}dV(\theta,\infty). \tag{7}$$

Second, suppose that $y^f(\tilde{s}(\theta), 0) > y^f(\theta, \infty)$, in which case the distorted shadow worker has a higher formal income that the distorted formal worker. In this case there is a discontinuity in the formal income schedule of the low-cost workers at $\tilde{s}(\theta)$. Denote by superscripts $\{-, +\}$ the directional limit of a given variable, e.g. $y^f(\tilde{s}(\theta)^-, 0)$ stands for the left limit of formal income of the low-cost workers at $\tilde{s}(\theta)$. From the definition of the mapping $\tilde{s}$ we know that $y^f(\tilde{s}(\theta)^-, 0) < y^f(\tilde{s}(\theta)^+, 0)$.

The perturbation of the formal income of type (θ, ∞) decreased the utility of all workers with formal income above $y^f(\theta, \infty)$, including $\tilde{s}(\theta)$, by $dV_\theta(\theta, \infty)$. It means that the perturbation, absent behavioral responses, leads to a discontinuity at $\tilde{s}(\theta)$ in the utility schedule of the low-cost workers, which is not incentive compatible. The behavioral responses will restore the continuity of $V(\theta, 0)$ by adjusting the mapping $\tilde{s}(\theta)$. Denote this adjustment by $d\tilde{s}(\theta)$.

Continuity of $V(\theta, 0)$ at $\tilde{s}(\theta)$ means that $V(\tilde{s}(\theta)^-, 0) = V(\tilde{s}(\theta)^+, 0)$. Suppose that the utility of worker $(\tilde{s}(\theta), 0)$ is decreased by dT. Continuity of the utility schedule requires that

$$V_\theta(\tilde{s}(\theta)^-,0)d\tilde{s}(\theta) = V_\theta(\tilde{s}(\theta)^+,0)d\tilde{s}(\theta) - dT$$
$$\implies d\tilde{s}(\theta) = \frac{w^f(\tilde{s}(\theta))/w^f(\tilde{s}(\theta))}{\rho^f(\tilde{s}(\theta)) - \rho^s(\tilde{s}(\theta))} \frac{dT}{y^f(\tilde{s}(\theta)^+,0) - y^f(\tilde{s}(\theta)^-,0)}.$$

This adjustment of $\tilde{s}(\theta)$ is associated with a tax loss

$$\Big(T(y^f(\tilde{s}(\theta)^+,0)) - T(y^f(\tilde{s}(\theta)^-,0))\Big) f(\tilde{s}(\theta))G_{\tilde{s}(\theta)}(\tilde{\kappa}(\theta))d\tilde{s}(\theta). \tag{8}$$

Note that $V(\tilde{s}(\theta)^-,0) = V(\tilde{s}(\theta)^+,0)$ implies that

$$\frac{T(\tilde{s}(\theta)^+,0) - T(\tilde{s}(\theta)^-,0)}{y^f(\tilde{s}(\theta)^+,0) - y^f(\tilde{s}(\theta)^-,0)} = 1 - \frac{w^s(\tilde{s}(\theta))}{w^f(\tilde{s}(\theta))}. \tag{9}$$

Using this result, we can express the tax loss as

$$\frac{w^f(\tilde{s}(\theta)) - w^s(\tilde{s}(\theta))}{w^s(\tilde{s}(\theta))} \frac{G_{\tilde{s}(\theta)}(\tilde{\kappa}(\tilde{s}(\theta)))f(\tilde{s}(\theta))}{\rho^f(\tilde{s}(\theta)) - \rho^s(\tilde{s}(\theta))} dT. \tag{10}$$

Notice the dT is equal to $dV_\theta(\theta,\infty)$. Hence, the tax loss is the same as in the previous case, when $y^f(\tilde{s}(\theta),0) = y^f(\theta,\infty)$.

Impact on workers with higher formal income. First, suppose that $y^f(\theta,\infty) \leq y^f(\bar{\theta},0)$. The perturbation implies a shift $dV_\theta(\theta,\kappa)$ in utility levels of formal workers above type θ and shadow workers above $\tilde{s}(\theta)$. Recall that the marginal social welfare weights are equal to the Pareto weights. The fiscal and welfare impact of such change is

$$\int_\theta^{\tilde{s}(\theta)} \int_{\tilde{\kappa}(\theta')}^\infty \big(\lambda(\theta',\kappa) - 1\big)\, dG(\kappa)dF(\theta')dV_\theta(\theta,\infty)$$
$$+ \int_{\tilde{s}(\theta)}^{\bar{\theta}} \int_0^\infty (\lambda(\theta',\kappa) - 1)dG(\kappa)dF(\theta')dV_\theta(\theta,\infty). \tag{11}$$

Note that among the productivity types in the segment $(\theta,\tilde{s}(\theta))$ the high-cost workers are affected by the perturbation, but the low-cost worker are not. Hence, the perturbation changes the threshold $\tilde{\kappa}$ at this segment. Denote by $\tilde{\Delta}T(\theta) \equiv T(y^f(\theta,\infty)) - T(y^f(\theta,0))$ the tax loss from worker of type θ moving to the shadow economy. The fiscal impact of the change in participation is

$$\int_\theta^{\tilde{s}(\theta)} \tilde{\Delta}T(\theta')g_{\theta'}(\tilde{\kappa}(\theta'))dF(\theta')dV_\theta(\theta,\infty). \tag{12}$$

In the case of $y^f(\theta,\infty) > y^f(\bar{\theta},0)$ only the formal workers are affected by a tax reform. The total fiscal and welfare impact on agents with higher formal income is

$$\int_{\theta}^{\bar{\theta}} \left[\int_{\tilde{\kappa}(\theta')}^{\infty} \big(\lambda(\theta',\kappa) - 1\big)\, dG(\kappa) + \tilde{\Delta} T(\theta') g_{\theta'}(\tilde{\kappa}(\theta')) \right] dF(\theta') dV_{\theta}(\theta,\infty). \tag{13}$$

Collecting the terms. At the optimum, the total impact of a small perturbation is zero. First, consider the case of $y^f(\theta,\infty) \leq y^f(\bar{\theta},0)$. The sum of the distortion cost of a high-cost worker (4), the distortion cost of the low-cost worker (7) as well as of impacts on the workers with higher formal income (11) and (12) needs to be zero, which results in

$$\begin{aligned}
\frac{T'(y^f(\theta,\infty))}{1-T'(y^f(\theta,\infty))} \frac{(1-G_{\theta}(\tilde{\kappa}(\theta)))f(\theta)}{\rho^f(\theta)(1+\varepsilon^{-1}(\theta,\infty))} &+ \frac{w^f(\tilde{s}(\theta)) - w^s(\tilde{s}(\theta))}{w^s(\tilde{s}(\theta))} \frac{G_{\tilde{s}(\theta)}(\tilde{\kappa}(\tilde{s}(\theta)) f(\tilde{s}(\theta))}{\rho^f(\tilde{s}(\theta)) - \rho^s(\tilde{s}(\theta))} \\
= \int_{\theta}^{\tilde{s}(\theta)} &\left[\int_{\tilde{\kappa}(\theta')}^{\infty} (1-\lambda(\theta',\kappa)) dG_{\theta'}(\kappa) - g_{\theta'}(\tilde{\kappa}(\theta'))\tilde{\Delta} T(\theta') \right] dF(\theta') \\
&+ \int_{\tilde{s}(\theta)}^{\overline{\theta}} \int_0^{\infty} (1-\lambda(\theta',\kappa)) dG_{\theta'}(\kappa) dF(\theta'). \qquad (14)
\end{aligned}$$

If the perturbation affects no shadow workers ($y^f(\theta,\infty) > y^f(\bar{\theta},0)$), the terms (4) and (13) should sum up to zero, which yields

$$\begin{aligned}
&\frac{T'(y^f(\theta,\infty))}{1-T'(y^f(\theta,\infty))} \frac{(1-G_{\theta}(\tilde{\kappa}(\theta)))f(\theta)}{\rho^f(\theta)(1+\varepsilon^{-1}(\theta,\infty))} \\
&\qquad = \int_{\theta}^{\overline{\theta}} \left[\int_{\tilde{\kappa}(\theta')}^{\infty} \big(1-\lambda(\theta',\kappa)\big)\, dG_{\theta'}(\kappa) - g_{\theta'}(\tilde{\kappa}(\theta'))\tilde{\Delta} T(\theta') \right] dF(\theta'). \qquad (15)
\end{aligned}$$

A.2. Definitions of sufficient statistics

$\varepsilon^x(\theta)$ and $\tilde{\varepsilon}^x(\theta)$ stand for the formal income elasticity of workers in sector $x \in \{f,s\}$ with respect to the marginal tax rate along the linear and non-linear tax schedule, respectively. $\varepsilon^x_{w^f}(\theta)$ and $\tilde{\varepsilon}^x_{w^f}(\theta)$ stand for the formal income elasticity of workers in sector $x \in \{f,s\}$ with respect to the gross formal wage along the linear and non-linear tax schedule, respectively. The elasticities of formal workers are derived from the optimality condition $y^f(\theta,\infty) = w^f(\theta)(v')^{-1}\big((1-T'(y^f(\theta,\infty)))w^f(\theta)\big)$, while the elasticities of shadow workers are derived from the optimality condition $(1-T'(y^f(\theta,0)))w^f(\theta) = w^s(\theta)$.

The elasticities of formal workers are

$$\varepsilon^f(y^f(\theta,\infty)) \equiv \frac{v'(n(\theta,\infty))}{n(\theta,\infty)v''(n(\theta,\infty))}, \tag{16}$$

$$\tilde{\varepsilon}^f(y) \equiv \left[\frac{1}{\varepsilon^f(y)} + \frac{T''(y)y}{1-T'(y)}\right]^{-1}, \tag{17}$$

$$\varepsilon^f_{w^f}(y) \equiv 1 + \varepsilon^f(y), \tag{18}$$

$$\tilde{\varepsilon}^f_{w^f}(y) \equiv \frac{\tilde{\varepsilon}^f(y)}{\varepsilon^f(y)}\varepsilon^f_{w^f}(y). \tag{19}$$

The elasticities of shadow workers are

$$\tilde{\varepsilon}^s(y) \equiv \frac{1-T'(y)}{T''(y)y}, \tag{20}$$

$$\tilde{\varepsilon}^s_{w^f}(y^f(\theta,0)) \equiv \left(1 - \frac{\rho^s(\theta)}{\rho^f(\theta)}\right)\tilde{\varepsilon}^s(y^f(\theta,0)). \tag{21}$$

Note that shadow workers have infinite elasticities of formal income along the linear tax schedule: as soon as the net formal wage departs from the shadow wage, the shadow worker either stops supplying formal labor entirely or becomes a formal worker. Nevertheless, elasticities along the non-linear tax schedule are well defined, as long the tax schedule is not locally linear.

Denote the derivative of formal income w.r.t. the productivity type along the non-linear tax schedule as

$$\tilde{y}^f_\theta(\theta,\kappa) \equiv \begin{cases} \tilde{\varepsilon}^f_{w^f}(y^f(\theta,\kappa))\rho^f(\theta)y^f(\theta,\kappa) & \text{if } \kappa \geq \tilde{\kappa}(\theta), \\ \tilde{\varepsilon}^s_{w^f}(y^f(\theta,\kappa))\rho^f(\theta)y^f(\theta,\kappa) & \text{otherwise.} \end{cases} \tag{22}$$

The density of formal workers at formal income $y^f(\theta,\infty)$, scaled by the share of formal workers, is defined as $h^f(y^f(\theta,\infty)) \equiv (1 - G_\theta(\tilde{\kappa}(\theta)))f(\theta)/\tilde{y}^f_\theta(\theta,\infty)$. The density of shadow workers at formal income $y^f(\theta,0)$, scaled by the share of shadow workers, is $h^s(y^f(\theta,0)) \equiv G_\theta(\tilde{\kappa}(\theta))f(\theta)/\tilde{y}^f_\theta(\theta,0)$ and $h^s(y^f) \equiv 0$ for $y^f \notin y^f([\underline{\theta},\bar{\theta}],0)$. The density of formal income is simply $h(y) \equiv h^f(y) + h^s(y)$. The mean elasticity at income level y is $\bar{\varepsilon}(y) \equiv h^f(y)\tilde{\varepsilon}^f(y) + h^s(y)\tilde{\varepsilon}^s(y)$.

Define the elasticity of the density of formal workers with respect to the tax burden of staying formal $\tilde{\Delta}T(\theta)$ as

$$\pi(y^f(\theta,\infty)) \equiv \frac{g_\theta(\tilde{\kappa}(\theta))\tilde{\Delta}T(\theta)}{1 - G_\theta(\tilde{\kappa}(\theta))}. \tag{23}$$

Define the average welfare weights of formal and shadow workers at a given formal income as

$$\bar{\lambda}^f(y^f(\theta,\infty)) \equiv \int_{\tilde{\kappa}(\theta)}^{\infty} \lambda(\theta,\kappa)\frac{dG_\theta(\kappa)}{1-G_\theta(\tilde{\kappa}(\theta))}, \quad \bar{\lambda}^s(y^f(\theta,0)) \equiv \int_0^{\tilde{\kappa}(\theta)} \lambda(\theta,\kappa)\frac{dG_\theta(\kappa)}{G_\theta(\tilde{\kappa}(\theta))}. \quad (24)$$

Then the average welfare weight at formal income y is $\bar{\lambda}(y) \equiv \left(h^f(y)\bar{\lambda}^f(y) + h^s(y)\bar{\lambda}^s(y)\right)/h(y)$. Finally, the mapping $\theta \mapsto s(\theta)$ is defined as $s(y^f(\theta,\infty)) \equiv y^f(\tilde{s}(\theta),0)$.

A.3. The equivalence of the mechanism design approach and the sufficient statistics approach

By substituting the terms defined above, we can represent the left-hand sides of (14) and (15) as in the sufficient statistics formulas from Theorem 1. In addition, we can represent the right-hand side of (14) as

$$\int_{\theta}^{\bar{\theta}} \left[1-\bar{\lambda}^f(y^f(\theta',\infty))\right](1-G_{\theta'}(\tilde{\kappa}(\theta')))dF(\theta') + \int_{\tilde{s}(\theta)}^{\bar{\theta}} \left[1-\lambda^s(y^f(\theta',0))\right] G(\tilde{\kappa}(\theta'))dF(\theta')$$
$$- \int_{\theta}^{\tilde{s}(\theta)} \frac{g_{\theta'}(\tilde{\kappa}(\theta'))\tilde{\Delta}T(\theta')}{1-G_{\theta'}(\tilde{\kappa}(\theta'))}(1-G_{\theta'}(\tilde{\kappa}(\theta')))dF(\theta'). \quad (25)$$

By changing variables we obtain

$$\int_{y^f(\theta,\infty)}^{\infty} \left[1-\bar{\lambda}^f(y)\right] h^f(y)dy + \int_{y^f(\tilde{s}(\theta),0)}^{\infty} \left[1-\bar{\lambda}^s(y)\right] h^s(y)dy - \int_{y^f(\theta,\infty)}^{y^f(\tilde{s}(\theta),\infty)} \pi(y)h^f(y)dy \quad (26)$$

$$= \int_{y^f(\theta,\infty)}^{\infty} \left[1-\bar{\lambda}(y)\right] h(y)dy - \int_{y^f(\theta,\infty)}^{y^f(\tilde{s}(\theta),\infty)} \pi(y)h^f(y)dy. \quad (27)$$

Finally, note that $y^f(\tilde{s}(\theta),\infty) = s(y^f(\theta,\infty)) + \Delta_0(s(y^f(\theta,\infty)))$, which means that the above expression is equal the right-hand side of the first formula from Theorem 1. We can express the right-hand side of (15) as the right-hand side of the second formula from Theorem 1 in an analogous way.

B. Pareto efficiency test of the Colombian tax schedule

The top panel of Figure I shows the actual tax and transfer schedule in Colombia in 2013. The marginal tax rates are high at the bottom due to the phase-out of transfers, then drop to 22% — the rate of payroll taxation — and then increase as the progressive personal income tax starts at around \$22,000. The marginal tax rate reaches 38% around \$50,000 and the top tax rate of 43% applies to incomes above \$175,000.

We can test the efficiency of the actual Colombian tax by extracting Pareto weights which would rationalize it. If at any income level the average Pareto weights are negative, the tax system is inefficient and the government can increase tax revenue without reducing utility of any agent.[1] To extract the welfare weights, differentiate the first optimal tax formula from Theorem 1 to get

$$\bar{\lambda}(y) = \mathbb{E}(\bar{\lambda}) + \underbrace{\frac{\partial DWL(y)}{\partial y}\frac{1}{h(y)}}_{\text{intensive margin}} - \underbrace{\pi(y)\frac{h^f(y)}{h(y)}}_{\text{extensive margin}}, \tag{28}$$

where $DWL(\cdot)$ stands for the total deadweight loss, i.e. the left-hand side of the first tax formula from Theorem 1, evaluated at formal income level y. The mean Pareto weight at a given income level can be explained by three components. The first one is the average Pareto weight across all income levels, equal to 1. The second is the contribution of the intensive margin. The total deadweight loss, including both formal and shadow workers, increases faster at income levels associated with higher Pareto weights. That is because a higher $\bar{\lambda}(y)$ reduces the deadweight loss below y and does not affect the deadweight loss above y, which is implied by the optimal tax formula. The third component captures the extensive margin: a decision to participate in the shadow economy. Recall that $\pi(y)$ is the elasticity of the density of formal workers with respect to the tax burden of staying formal. The impact of extensive margin is similar to that of the Pareto weight: it implies a higher derivative of the deadweight loss. Hence, a higher $\pi(y)$ means that a smaller part of the increase of deadweight loss remains to be explained by social preferences.

The Pareto weights implicit in the actual tax schedule are presented in the bottom panel of Figure I. We find no evidence of negative Pareto weights — the Colombian tax schedule is Pareto efficient.[2] However, the implicit weights exhibit a peculiar pattern: they are much lower for workers with earnings close to the minimum wage than for workers with slightly higher earnings. For instance, formal workers earning \$13,000 annually have an implied Pareto weight which is seven times smaller than the weight of workers earning \$19,000. Although the income interval of unusually low Pareto weights is relatively small, it contains 28% of all formal workers. None of the workers with formal earnings in this interval has shadow earnings.

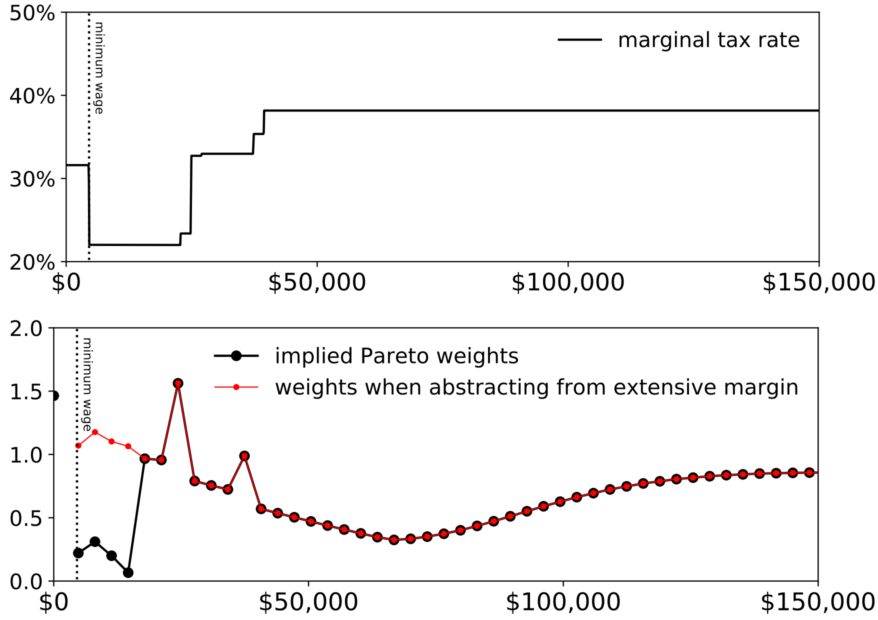
The pattern of unusually low Pareto weights at low incomes are driven by the extensive margin term in formula (28). If instead we ignored the extensive margin term, the Pareto weights would be more regular and decreasing with income at low income levels.[3] A possible interpretation of this result is that the Colombian tax schedule was

[1] The original test of Pareto efficiency was proposed by Werning (2007). The methodology was further developed and applied by Bourguignon and Spadaro (2012); Brendon (2013); Lorenz and Sachs (2016) and Jacobs, Jongen, and Zoutman (2017), among others.

[2] The actual tax schedule is efficient conditional on the value of the minimum wage. Since our framework is not designed to study the minimum wage, we do not evaluate its efficiency.

[3] Pareto weights are also locally increasing when marginal rates of the personal income tax are increasing rapidly. It is unrelated to the informal sector.

Figure I: Income tax schedule in Colombia and the implied Pareto weights



set without taking into account the extensive margin incentives for informality. How would accounting for these incentives modify the tax schedule? The actual tax schedule features costant marginal rates in the region where the Pareto weights are unusually low. The tax schedule which accounts for informality and follows the a intuitive, decreasing pattern of Pareto weights would instead have increasing, rather than flat, marginal rates in this region.

C. Welfare decomposition in a simple model

In this section we decompose the welfare impact of the existence of the shadow economy into the redistributive impact and efficiency impact. We consider a simplified version of the full model which allows us to characterize analytically comparative statics of both components. Specifically, we consider an economy with two types of workers, no fixed cost of shadow employment and no possibility of working simultaneously in the two sectors.

There are two types of individuals, indexed by L and H, with population shares μ_L and $\mu_H = 1 - \mu_L$. They care about consumption c and labor supply n according to a quasilinear utility function $U(c, n) \equiv c - v(n)$, where v is increasing, strictly convex, twice differentiable and satisfies $v'(0) = 0$. While the assumption of the linear utility from consumption allows for an easy exposition, it is straightforward to generalize the results from this section to concave utilities from consumption.

There are two labor markets and, correspondingly, each agent is equipped with two linear production technologies. An agent of type $i \in \{L, H\}$ produces with productivity w_i^f in a formal labor market and with productivity w_i^s in an informal labor market. Income in each sector is given by $y_i^x = w_i^x n_i^x$, where n_i^x denotes labor supply in sector $x \in \{f, s\}$. We identify type H as the one with higher formal productivity: $w_H^f > w_L^f$. Moreover, in this section we assume that each type is more productive formally: $\forall_i\ w_i^f > w_i^s$. It implies that the shadow economy is inefficient and is never used in the first-best when the planner can observe individual types.

C.1. The planner's problem

The social planner observes only the formal income of each individual. Furthermore, the planner can transfer resources between agents with taxes T_i. We can think about y_i^f and $y_i^f - T_i$ as a pre-tax and an after-tax reported income. It is convenient to express agents' choices of shadow income as a function of their formal income:

$$y_i^s\left(y^f\right) = w_i^s v'^{-1}(w_i^s) \times \mathbb{1}_{y^f=0}. \tag{29}$$

If agents have any formal earnings, their shadow earnings are zero. If instead they have no formal earnings, they are unconstrained in choosing their shadow income. Given this function, we can specify agents' consumption $c_i = y_i^f + y_i^s\left(y_i^f\right) - T_i$ and labor supply $n_i = y_i^f / w_i^f + y_i^s\left(y_i^f\right) / w_i^s$, conditional on a truthful revelation of types.

The social planner maximizes the sum of utilities weighted with Pareto weights λ_i

$$W = \max_{\left\{\left(y_i^f, T_i\right) \in \mathbb{R}_+ \times \mathbb{R}\right\}_{i\in\{L,H\}}} \lambda_L \mu_L U\left(c_L, n_L\right) + \lambda_H \mu_H U\left(c_H, n_H\right) \tag{30}$$

subject to a resource constraint

$$\mu_L T_L + \mu_H T_H \geq 0, \tag{31}$$

and incentive-compatibility constraints

$$U\left(c_i, n_i\right) \geq U\left(y_{-i}^f + y_i^s\left(y_{-i}^f\right) - T_{-i}, \frac{y_{-i}^f}{w_i^f} + \frac{y_i^s\left(y_{-i}^f\right)}{w_i^s}\right) \qquad i \in \{H, L\}. \tag{32}$$

The incentive compatibility constraints capture the limited information available to the planner. They imply that no agent can be better off by choosing formal income of the other type and, if this income level is zero, freely adjusting shadow earnings.

Lemma C.1. *Suppose that $\lambda_i > \lambda_{-i}$. In the optimum type i faces labor distortions and may work in the shadow economy, while type $-i$ faces no labor distortions and does not work in the shadow economy.*

Proof. The planner can increase social welfare by transferring consumption from type $-i$ to type i, so at the optimum the incentive constraint of $-i$ will binds and the incentive constraint of i will be slack. Denote the undistorted level of formal income of type $-i$ by $y_{-i}^{f*} = w_{-i}^f \cdot v'^{-1}(w_{-i}^f)$. If $y_{-i}^f \neq y_{-i}^{f*}$, the planner can extract more resources without violating the incentive constraint by setting $y_{i-}^f = y_{-i}^{f*}$ and increasing T_{-i} to keep the utility level of type $-i$ constant. Since $y_{-i}^{f*} > 0$, type $-i$ will not work in the shadow economy.

To see that the planner optimally distorts the labor supply of type i, notice that a marginal adjustment of y_i^f, starting from the undistorted level y_i^{f*}, has no direct impact on welfare of type i by the Envelope Theorem. However, the distortion in a correct direction will reduce the payoff of $-i$ from misreporting, relax the incentive constraint and, hence, allow for more redistribution. In particular, if $w_i^f < w_{-i}^f$ ($w_i^f > w_{-i}^f$), a marginal decrease (increase) of y_i^f will relax the incentive constraint. □

Lemma C.1 is a generalization of the classic *no distortion at the top* result. When $\lambda_i > \lambda_{-i}$, the planner wants to redistribute from type $-i$ to type i. The incentive constraint of type $-i$ will bind, and hence the planner cannot improve the allocation by distorting labor of type $-i$. Since an agent works in the shadow economy only if his formal labor is sufficiently distorted downwards (and equal to zero), the agent $-i$ will never work in the shadow economy in the optimum. On the other hand, distorting the labor choice of type i relaxes the binding incentive constraint and allows for more redistribution. Hence, type i can potentially work in the shadow economy in the optimum.

C.2. Welfare decomposition

Suppose that $\lambda_i > \lambda_{-i}$, such that the planner wants to redistribute resources from type $-i$ to type i. There are two candidate allocations for the optimum: a *Mirrleesian allocation* in which type i works formally (denoted with superscript M) and a *shadow economy allocation* in which type i works informally (denoted with superscript SE). Note that the *Mirrleesian allocation* is also the optimum in the setting without the shadow economy. We examine the welfare impact of the shadow economy by comparing these two allocations.

Proposition 1. *Suppose that $\lambda_i > \lambda_{-i}$. The welfare difference between the shadow economy allocation and the Mirrleesian allocation can be decomposed in the following way*

$$\underbrace{W^{SE} - W^M}_{\textit{welfare impact}} = \underbrace{\lambda_i \mu_i \left(U\left(w_i^s n_i^{SE}, n_i^{SE}\right) - U\left(w_i^f n_i^M, n_i^M\right)\right)}_{\textit{efficiency impact}} + \underbrace{(\lambda_i - \lambda_{-i})\mu_i \left(T_i^M - T_i^{SE}\right)}_{\textit{redistributive impact}},$$

where

- *the efficiency impact is increasing with* w_i^s *and is positive when* $w_i^s > \bar{w}_i^s$,
- *the redistributive impact is decreasing with* w_{-i}^s *and is positive when* $w_{-i}^s < \bar{w}_{-i}^s$,
- *the productivity thresholds satisfy* $\bar{w}_i^s < w_i^f$ *and* $\bar{w}_{-i}^s < w_{-i}^f$.

Proof. The difference in the utility level of type i between the two allocations is

$$U\left(c_i^{SE}, n_i^{SE}\right) - U\left(c_i^M, n_i^M\right) = U\left(w_i^s n_i^{SE}, n_i^{SE}\right) - U\left(w_i^f n_i^M, n_i^M\right) + T_i^M - T_i^{SE}. \quad (33)$$

The difference in utility level of type $-i$ is

$$U\left(c_{-i}^{SE}, n_{-i}^{SE}\right) - U\left(c_{-i}^M, n_{-i}^M\right) = T_{-i}^M - T_{-i}^{SE} = -\frac{\mu_i}{\mu_{-i}}\left(T_i^M - T_i^{SE}\right), \quad (34)$$

where the first equality follows from Lemma C.1, since in the two allocations the labor supply of $-i$ is undistorted, and the second equality follows from the resource constraint. Using both utility differences, we can decompose $W^{SE} - W^M$ as stated in the proposition.

Define a function $\Psi(w) = U\left(wv'^{-1}(w), v'^{-1}(w)\right)$, equal to the utility level of an individual with productivity w who supplies labor efficiently and receives no transfers. The efficiency impact can be restated as $\lambda_i \mu_i \left(\Psi(w_i^s) - U\left(w_i^f n_i^M, n_i^M\right)\right)$. Since Ψ is an increasing function, the efficiency impact is increasing in w_i^s and changes sign at $\bar{w}_i^s = \Psi^{-1}\left(U\left(w_i^f n_i^M, n_i^M\right)\right)$. To see that $\bar{w}_i^s < w_i^f$, note that since n_i^M is distorted, $\Psi(w_i^f) > U\left(w_i^f n_i^M, n_i^M\right)$.

To characterize the redistributive impact, note that, due to the binding incentive constraints, we have

$$U\left(c_{-i}^{SE}, n_{-i}^{SE}\right) - U\left(c_{-i}^M, n_{-i}^M\right) = \Psi(w_{-i}^s) - T_i^{SE} - U\left(w_i^f n_i^M, w_i^f n_i^M / w_{-i}^f\right) + T_i^M. \quad (35)$$

Combining it with (34), we find that $T_i^M - T_i^{SE} = \mu_{-i}\left(U\left(w_i^f n_i^M, w_i^f n_i^M / w_{-i}^f\right) - \Psi(w_{-i}^s)\right)$. It implies that the redistributive impact is decreasing in w_{-i}^s and changes sign at $\bar{w}_{-i}^s = \Psi^{-1}\left(U\left(w_i^f n_i^M, w_i^f n_i^M / w_{-i}^f\right)\right)$. The inequality $\bar{w}_{-i}^s < w_{-i}^f$ holds since $U\left(w_i^f n_i^M, w_i^f n_i^M / w_{-i}^f\right) < \Psi(w_{-i}^f)$ due to the optimal distortion of n_i^M. □

Proposition 1 decomposes the welfare impact of the shadow economy into an *efficiency impact*, measuring the difference in distortions imposed on type i, and a *redistributive impact*, capturing the change in the level of transfers received by type i.

Efficiency impact. In the shadow economy allocation, type i supplies the efficient level of labor to the inefficient shadow sector. In the Mirrleesian allocation, due to the distortions imposed by the planner, type i supplies an inefficient amount of labor to the efficient formal sector. The relative inefficiency of the shadow sector depends on the productivity difference $w_i^f - w_i^s$. When this difference is sufficiently small ($w_i^s > \bar{w}_i^s$), distortions in the shadow sector are smaller than distortions in the formal sector and the shadow economy improves the efficiency of labor allocation. Intuitively, in this case the shadow economy provides a shelter against tax distortions. If instead the shadow economy distortions are large ($w_i^s < \bar{w}_i^s$), the efficiency impact of the informal sector will be negative.

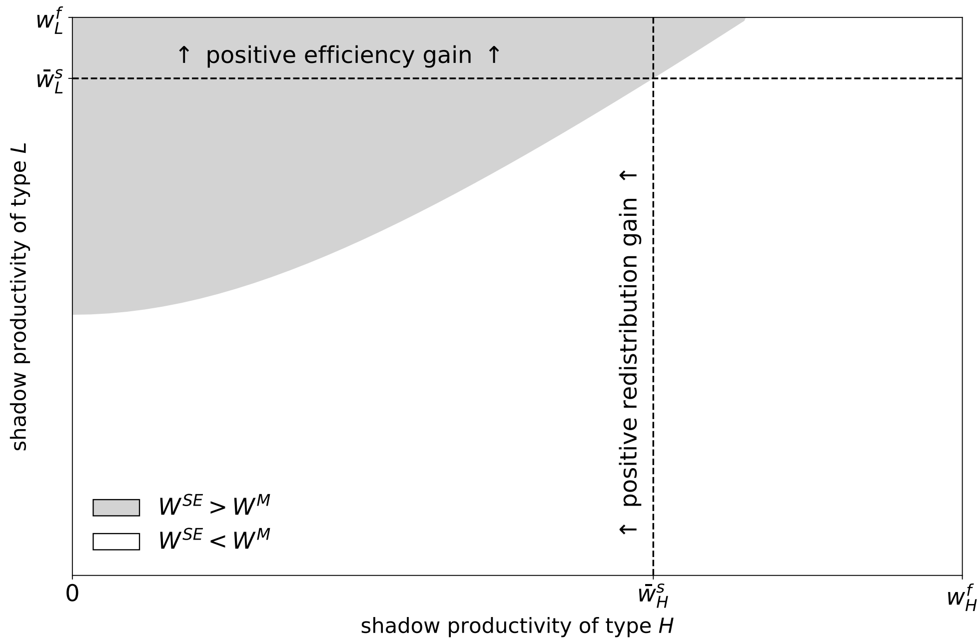
Redistributive impact. The shadow economy improves redistribution if the planner is able to provide type i with a higher transfer (or equivalently raise a higher tax from type $-i$). The scale of redistribution is determined by the payoff of type $-i$ from misreporting. In the Mirrleesian allocation the deviating agent works formally and can earn only as much as type i. In the shadow economy allocation the deviating worker cannot supply any formal labor, but is unconstrained in supplying shadow labor. As the shadow productivity of type $-i$ increases, the payoff from misreporting in the shadow economy allocation rises and the redistribution is reduced. On the other hand, when w_{-i}^s is sufficiently low ($w_{-i}^s < \bar{w}_{-i}^s$), the shadow economy deters the deviation of type $-i$, helping the planner to tell the two types of agents apart. In this case the informal sector is effectively used as a screening device.

Proposition 1 is illustrated in Figure III, where we assume that the planner maximizes the utility of type L: $\lambda_H = 0$. Intuitively, the shadow economy does not have to strengthen both redistribution and efficiency simultaneously to be welfare improving. Particularly interesting is the region where the redistributive impact is negative, but the efficiency impact is sufficiently high such that welfare is higher with the shadow economy. In this case the shadow economy allocation Pareto dominates the Mirrleesian allocation. Type L gains, since the welfare is higher with the shadow economy. Type H benefits as well, as the negative redistribution gain implies a lower tax burden on this type.

Kopczuk (2001) provides an example in which, starting from the allocation without tax avoidance, a marginal increase in evasion yields welfare gains.[4] According to our decomposition, in his example welfare improves due to greater redistribution, but at the cost of efficiency. It may suggest that the shadow economy can improve welfare by allowing for more even division of a smaller aggregate output. Our results show that such scenario is only one of many possibilities. For instance, the shadow economy can reduce redistribution, while still being welfare-improving, in which case all agents benefit from the presence of the shadow economy.

[4] Kopczuk (2001) also presents a second example of welfare-improving tax avoidance in which some agents have a distaste for paying taxes. We abstract from agents having preferences directly over tax payments.

Figure III: Welfare impact of the shadow economy



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